Gravitational force in weakly correlated particle spatial distributions

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We study the statistics of the gravitational (Newtonian) force in a particular class of weakly correlated spatial distributions of pointlike and unitary mass particles generated by the so-called Gauss-Poisson point processes. In particular we extend to these distributions the analysis that Chandrasekhar introduced for purely Poisson processes. In this way we can find the explicit asymptotic behavior of the probability density function of the force for both large and small values of the field as a generalization of the Holtzmark statistics. In particular, we show how the modifications at large fields depend on the density correlations introduced at small scales. The validity of the introduced approximations is positively tested through a direct comparison with the analysis of the statistics of the gravitational force in numerical simulations of Gauss-Poisson processes.

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I. INTRODUCTION

The knowledge of the statistical properties of the gravitational field in a given spatial distribution of point particles provides an important information about the system in many cosmological and astrophysical applications in which the interacting masses are treated as pointlike particles. In fact such information acquires particular importance in contexts such as the stellar dynamics and the cosmological *N*-body simulations for the study of the formation of structures from initial mass density perturbations at least at the "granular" scale [1,2]. Similar studies are involved in other domains of physics, such as the statistical physics of the *dislocation-dislocation* interaction for what concerns the analysis of crystal defects in condensed matter physics [3].

Until now a complete study of this problem has been accomplished only in the case of uncorrelated particle spatial distributions obtained by Poisson point processes [4,5]. Partial results have been found more recently in two other cases: (1) a fractal particle distribution [6], and (2) a radial density of particles [7]. In this paper we study the case of the so-called Gauss-Poisson (GP) class of point processes [8,9] which generates particle spatial distributions characterized only by short-range two-point correlations, i.e., connected n-point correlation functions are integrable for $n=2$ and vanish for $n\geqslant 3$. In this sense it can be seen as the first step of correlated systems beyond the completely uncorrelated Poisson point process, and is characterized spatially by the presence of binary systems. For this last reason the introduction of such a class of point processes is useful to study the statistical physics of many spatial distributions of particles characterized mainly by such binary structures.

First of all we recall the main rigorous results about the probability density function of the gravitational force in a particle system generated by a pure homogeneous and isotropic Poisson point process as obtained by Chandrasekhar in Ref. [5]. For the GP class of point processes we generalize these methods introducing appropriate approximations in order to exploit, in this framework, as much as possible the information encrypted in the two-point density-density correlation function. In this way an integral form for the conditional probability density function of the gravitational force is obtained, and explicit scaling laws for both large and small values of the force are given for any spatial distribution of point particles belonging to the GP class. The validity of these theoretical results is confirmed by direct comparison with those obtained by the statistical analysis of numerical simulations of exact GP particle distributions.

II. GRAVITATIONAL FORCE PROBABILITY DENSITY IN A POISSON POINT PROCESS

First, let us recall Chandrasekhar's [5] results for the Poisson case. A homogeneous and isotropic Poisson spatial distribution of pointparticles with average number density n in a volume V can be obtained as follows.

(1) Partition the space in cells of volume dV.

(2) Occupy randomly with a particle of unit mass each of these cells with probability $n\,dV$ (with $n>0$ and $n\,dV\ll 1$) or leave it empty with complementary probability $1-n\,dV$ with no correlation between different cells.

From this definition it is simple to find that the average number of particles in a box of volume V is simply $\langle N\rangle = nV$, and that fluctuations from realization to realization of this number are of the order of $\sqrt{nV}$ (the so-called Poisson or white noise), which become negligibly small with respect to $\langle N\rangle$ in the large V limit. In general, the internal spatial correlations of a stochastic spatial distribution of unit mass particles are measured by the connected two-point correlation function (CTPCF) $\xi(\vec{x})$ defined by

$$\xi(\vec{x}) = \frac{\langle n(\vec{x}_0) n(\vec{x}_0 + \vec{x}) \rangle}{n^2} - 1,$$

where $n(\vec{x}) = \Sigma_i \delta(\vec{x} - \vec{x}_i)$ is the microscopic number density field in which the sum is extended to all the particles position $\vec{x}_i$ and $\langle \cdots \rangle$ is the ensemble average (or volume average in the infinite volume limit in case of ergodicity, as in all point processes treated here). Since in the definition of the Poisson case there is no correlation between the probability of occupations of different cells, it is simple to show that the CTPCF has only the diagonal part due to granularity, i.e.,

$$\xi(\vec{x}) = \delta(\vec{x})/n, \tag{1}$$

meaning that each particle is spatially correlated only with itself. Any other statistically homogeneous particle distribution with a well defined average density $n > 0$ is characterized by a CTPCF of the form [10,11]

$$\xi(\vec{x}) = \frac{\delta(\vec{x})}{n} + h(\vec{x}), \tag{2}$$

where $h(\vec{x})$ is the nondiagonal part due to correlations between the positions of different particles.

Fixing arbitrarily all the physical constants equal to one, the gravitational field acting on the origin of axis is given by

$$\vec{F} = \sum_i \frac{\vec{x}_i}{x_i^3}, \tag{3}$$

where the sum runs over all the system particles out of the origin. If the origin of axes is occupied by a system particle, Eq. (3) gives the gravitational force experienced by it. Once the statistical ensemble of spatial distributions of particles is chosen, it is possible to evaluate the probability density function (PDF) $P(\vec{F})$ of the field $\vec{F}$ by taking the average of $\delta(\vec{F} - \Sigma_i \vec{x}_i / x_i^3)$ over the ensemble conditioned to the fact that the origin is occupied by a system particle. For the above introduced Poisson system in a volume V with average density of particles n this calculation can be performed exactly [5] by taking into account that in this case the stochastic positions of different particles are completely uncorrelated, and that spatially the system is statistically homogeneous and isotropic. Consequently, the joint PDF of the positions of the N particles out of the origin, and conditioned to having the origin occupied by another particle, can be written as

$$p_c(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N) = \prod_{i=1}^{N} \frac{1}{V} = \frac{1}{V^N}, \tag{4}$$

which in this case is identical also to the unconditional joint PDF. These considerations permit to write:

$$P(\vec{F}) = \int_V \cdots \int_V \left[\prod_{i=1}^{N} \frac{d^3 x_i}{V} \right] \delta\left(\vec{F} - \sum_{i=1}^{N} \frac{\vec{x}_i}{x_i^3} \right).$$

By using the Fourier representation of the Dirac delta function and taking the infinite volume limit $V \to +\infty$ with $N/V = n$ fixed (fluctuations of N from the average value nV can be shown to give vanishing corrections in the infinite volume limit), we obtain:

$$P(\vec{F}) = \frac{1}{(2\pi)^3} \int d^3k e^{i\vec{k}\cdot\vec{F} - nC_P(k)}, \tag{5}$$

where

$$C_P(k) = \frac{4}{15}(2\pi k)^{3/2}. \tag{6}$$

Note that $A(k) = \exp[-nC_P(k)]$ is the Fourier transform of $P(\vec{F})$, i.e., $A(k)$ is the so-called *characteristic function* of the stochastic force $\vec{F}$ [4]. Since $C_P(k)$ depends only on $k = |\vec{k}|$, $P(\vec{F})$ will depend only on $F = |\vec{F}| \geq 0$. That is, the direction (θ, ϕ) of $\vec{F}$ is completely random and statistically the modulus F is distributed following the PDF $W(F) = 4\pi F^2 P(\vec{F})$ that can be rewritten as

$$W(F) = \frac{2F}{\pi} \int_0^\infty dk k \sin(kF) \exp\left(-\frac{4n}{15}(2\pi k)^{3/2} \right). \tag{7}$$

This important result is known under the name of *Holtzmark* PDF (for a general account of the Holtzmark PDF and other stable probability distributions and their expression in terms of special functions see Refs. [12,13]). An explicit expression of $W(F)$ is not obtainable; anyway it is rather simple to study the asymptotic regimes for small and large values of F [5]:

$$W(F) \simeq \begin{cases} \frac{4}{3\pi} F_0^{-3} F^2 & \text{for } F \to 0^+ \\ \frac{4\sqrt{2\pi}}{15} F_0^{3/2} F^{-5/2} = 2\pi n F^{-5/2} & \text{for} F \to \infty, \end{cases} \tag{8}$$

where

$$F_0 = 2\pi \left(\frac{4n}{15} \right)^{2/3} \tag{9}$$

That is, Eq. (8) gives the two asymptotic behaviors, respectively, for $F \ll n^{2/3}$ and $F \gg n^{2/3}$ roughly, where, $1/n$ being the average volume per particle, $n^{2/3}$ gives the order of the nearest particles interaction.

Now we show that the limit behavior for large F is mainly determined by the position of the first nearest neighbor (NN) particle. In order to show this result in more detail we have to evaluate the probability $\omega(x)dx$ that, given a particle, its first NN is at a distance between x and $x + dx$ from it. Considering that the probability of finding the NN particle between x and $x + dx$ is equal to the product of the probability that there is no particle in the distance interval $(0, x]$ and the probability $4\pi n x^2 dx$ of finding a generic particle in the interval of distances $(x, x + dx]$ [14], we can write

$$\omega(x)=\left(1-\int_0^x \omega(x)dx\right)4\pi x^2 n. \tag{10}$$

The derivation of Eq. (10) is based on the fact that for a Poisson point process there is no correlation between the position of different particles. This implies that the probability of finding no particle in $(0,x]$ is independent of the probability of finding a particle in $(x,x+dx]$. This of course holds for the homogeneous Poisson case, but in general it is not true for spatially correlated point processes. Equation (10) can be simply solved to give:

$$\omega(x)=4\pi n x^2 \exp\left(-\frac{4\pi}{3}nx^3\right). \tag{11}$$

By considering that the force exerted by the NN particle is $f=1/x^2$, we can find, by a simple change of variable, the PDF of the modulus of the gravitational field generated by the first neighbor as

$$W_{nn}(f)=2\pi n f^{-5/2}\exp\left(-\frac{4\pi n f^{-3/2}}{3}\right). \tag{12}$$

In the limit $f\gg n^{2/3}$ Eq. (12) reads

$$W_{nn}(f)\simeq 2\pi n f^{-5/2}, \tag{13}$$

which is exactly the same asymptotic behavior of the PDF $W(F)$ of the modulus of the total force found in Eq. (8). This means that in a Poisson spatial distribution of particles, the main contribution to the force acting on one of them comes from the other particles in its neighborhood, implying that the force fluctuates a lot in space from particle to particle.

III. THE GAUSS-POISSON POINT PROCESS

We now discuss the one-point statistical properties of the gravitational Newtonian field in a well defined class of weakly correlated particle systems. In particular, we analyze the spatial distributions of pointlike field sources (of unit mass) generated by GP point processes (briefly GP particle distributions). A GP particle distribution [8,9] is built in the following way.

(1) Take a statistically homogeneous and isotropic Poisson spatial distribution of particles of average density $n_0>0$;

(2) The next step is to pick up randomly a fraction $0<q\leq 1$ of these Poisson points called "parents" and attach to each of them a new "daughter" particle in the volume element d^3x at vectorial distance $\vec{x}$ from the "parent" particle with probability $p(\vec{x})d^3x$, each parent independent of the others. The function $p(\vec{x})$ is the PDF of the vectorial distance $\vec{x}$ of attachment and clearly is integrable and normalized. In what follows we will suppose that $p(-\vec{x})=p(\vec{x})$.

The net effect of this algorithm is of substituting a fraction q of particles of the initial Poisson system with an equal number of correlated binary systems. This is the reason why this kind of point process can be very useful in all the physical applications characterized by the presence of binary systems.

It is immediate to show that the final particle density in the so-generated GP particle distribution is $n=n_0(1+q)$. It is also possible to show that the CTPCF is

$$\xi(\vec{x})=\frac{\delta(\vec{x})}{n}+\frac{2q}{n(1+q)}p(\vec{x}), \tag{14}$$

and that all the other connected l-point correlation functions with $l\geq 3$ vanish [15]. This means that all the statistics of a GP point process is reduced to the knowledge of n and $\xi(\vec{x})$. For this reason the GP particle distribution is the discrete analog of the continuous Gaussian stochastic field. Moreover, since $p(\vec{x})$ is a PDF, $\xi(\vec{x})$ is non-negative and integrable over all the space, i.e., spatial correlations are positive and short ranged. This is the reason why the GP point process can be seen as the most weakly correlated particle system beyond the Poisson one.

To show the validity of Eq. (14) is a quite simple task. This is done by using the definition of average *conditional* density $n_c(\vec{x})$ of particles seen by a generic particle of the system at a vectorial distance $\vec{x}$ from it in terms of the nondiagonal part $h(\vec{x})$ of the CTPCF [11]:

$$n_c(\vec{x})=n[1+h(\vec{x})]. \tag{15}$$

In the GP model $n_c(\vec{x})$ can be evaluated in the following way: the number of particles seen in average by a fixed particle in the volume element d^3x, around the vectorial distance $\vec{x}$ from it, is nd^3x if the chosen particle is neither a parent nor a daughter (i.e., with probability $(1-q)/(1+q)$) and $[n+p(\vec{x})]d^3x$ if it is either a parent or a daughter [i.e., with a complementary probability $2q/(1+q)$]. By averaging the two possibilities with the right weights, we have

$$n_c(\vec{x})=n\left[1+\frac{2q}{n(1+q)}p(\vec{x})\right],$$

which is equivalent to Eq. (14). Note that if $p(\vec{x})$ depends only on x (i.e., it is spherically symmetric), then the particle distribution, in addition to being statistically homogeneous (i.e., translational invariant), is also statistically isotropic.

It is worth noting that, as $p(\vec{x})$ is by definition integrable with unit integral, typical fluctuations of the number N of particles generated in single realizations of the GP process in a sufficiently large volume V (with n and q fixed) with respect to the average value $\langle N\rangle=nV$ is of the order of $\sqrt{nV}$, as in the Poisson case of the same average density n (but with a larger prefactor), which is very small with respect to nV in the large V limit. Consequently, in all the following calculations we will use, as in the Poisson case, directly $N=nV$, the correction due to fluctuations vanishing in the infinite volume limit.

IV. GENERALIZATION OF THE HOLTZMARK STATISTICS TO GAUSS-POISSON POINT PROCESSES

We now try to generalize the Holtzmark PDF to this weakly correlated case. Let us suppose of having a GP particle distribution with fixed $n>0$ and $0<q\leqslant 1$ in a volume V. As in the isotropic Poisson case let us also set the coordinate system in such a way that the origin is occupied by a particle of the system. We want to calculate the PDF $P(\vec{F})$ of the total gravitational field $\vec{F}$ acting on the origin of coordinates due to all the particles out of the origin conditioned to the fact that this point is occupied by a particle. Therefore, if the particles seen by the one in the origin are N, and $p_c(\vec{x}_1,\vec{x}_2,\dots,\vec{x}_N)$ is the joint conditional PDF of their positions, we can write

$$P(\vec{F})=\int_V\cdots\int_V\left[\prod_{i=1}^{N}d^3x_i\right]p_c(\vec{x}_1,\vec{x}_2,\dots,\vec{x}_N)\cdot\delta\times\left(\vec{F}-\sum_{i=1}^{N}\frac{\vec{x}_i}{x_i^3}\right) \quad (16)$$

Since in any GP point process, two-point correlations are present, $p_c(\vec{x}_1,\vec{x}_2,\dots,\vec{x}_N)$ cannot be written rigorously as a product of N one-particle PDF's as in the Poisson case. This feature would prevent us from applying the method used in the preceding section for the Poisson point processes, and an explicit evaluation of $P(\vec{F})$ would then become impossible. For this reason we introduce an approximation consisting in imposing the factorization

$$p_c(\vec{x}_1,\vec{x}_2,\dots,\vec{x}_N)=\prod_{i=1}^{N}\tau(\vec{x}_i) \quad (17)$$

in the best possible way. This is done by taking into account that, as $h(\vec{x})$ is short-ranged [being proportional to the integrable function $p(\vec{x})$] and the higher-order connected correlation functions vanish, we can limit ourselves to use the only information about the conditional density $n_c(\vec{x})$ around the occupied origin. As a matter of fact, directly from the definition of GP point processes, and the fact that the CTPCF is short ranged and of small amplitude, we can say that taking an arbitrary particle of such a system, it will see at enough large distance from it a Poisson (i.e., uncorrelated) particle distribution of particles of average density n, and at short scale, where $h(\vec{x})$ is appreciable, an almost radial particle distribution of inhomogeneous density $n_c(\vec{x})=n[1+h(\vec{x})]$. This leads to having $\tau(\vec{x})$ proportional to $n_c(\vec{x})$:

$$\tau(\vec{x})=\frac{1+\dfrac{2q}{n(1+q)}p(\vec{x})}{V+\dfrac{2q}{n(1+q)}}. \quad (18)$$

Note that this is equivalent to approximating the spatial distribution of N particles generated by the given statistically homogeneous GP point process with a statistically inhomogeneous and radial Poisson particle distribution generated by the following algorithm: once the space is partitioned in cells of volume dV, the cell around the point $\vec{x}$ is occupied with probability $n_c(\vec{x})dV$ [dV must be chosen in such a way that $n_c(\vec{x})dV\ll 1$] or stays unoccupied with the complementary probability $1-n_c(\vec{x})dV$ independent of the other cells.

Approximations (17) and (18) and the condition $N=nV$ permit us to use the method introduced in the preceding section to find $P(\vec{F})$ which, in the limit $V\to+\infty$, can be shown to be given by

$$P(\vec{F})=\frac{1}{(2\pi)^3}\int d^3k\,\exp[i\vec{k}\cdot\vec{F}-nC_{GP}(\vec{k})], \quad (19)$$

where

$$C_{GP}(\vec{k})=C_P(k)+\frac{2q}{n(1+q)}\int d^3x\,p(\vec{x})[1-e^{-i\vec{k}\cdot\vec{x}/x^3}], \quad (20)$$

with $C_P(k)$ given by Eq. (6). We sketch now the main steps to find Eqs. (19) and (20) using the approximations given by Eqs. (17) and (18). Starting from Eq. (16) with the approximation (17), and using the Fourier representation of the Dirac function we can write

$$P(\vec{F})=\frac{1}{(2\pi)^3}\int d^3k e^{i\vec{k}\cdot\vec{F}}\left(\int_V d^3x\,\tau(\vec{x})e^{-i\vec{k}\cdot\vec{x}/x^3}\right)^N. \quad (21)$$

By using the fact that $\int_V d^3x\,\tau(\vec{x})=1$, in the previous equation we can make the substitution

$$\int_V d^3x\,\tau(\vec{x})e^{-i\vec{k}\cdot\frac{\vec{x}}{x^3}}=1-\int_V d^3x\,\tau(\vec{x})\left(1-e^{-i\vec{k}\cdot\frac{\vec{x}}{x^3}}\right),$$

with $\tau(\vec{x})$ given by Eq. (18). Then we put $N=nV$ taking the limit $V\to\infty$ with n fixed, for which we use the mathematical definition of the exponential

$$e^A=\lim_{V\to+\infty}\left(1+\frac{A}{V}\right)^V.$$

As shown below by a direct comparison with the results of numerical simulations, this approximation is quite accurate at least in both the large and the small F limits. The function $A(\vec{k})=\exp[-nC_{GP}(\vec{k})]$ is the approximated *characteristic* function of the total stochastic force $\vec{F}$ acting on the particle in the origin in the GP case. As aforementioned, if the PDF $p(\vec{x})$ depends only on $x=|\vec{x}|$, the particle distribution is statistically isotropic and $P(\vec{F})$ will depend only on $F=|\vec{F}|$ and $A(\vec{k})$ on $k=|\vec{k}|$. That is, the direction of $\vec{F}$ is completely random while the PDF of F is given by [rewriting $p(\vec{x})$ as $p(x)$ to exhibit the dependence only on x]

$$W(F)\equiv 4\pi F^2 P(\vec{F}) = \frac{2F}{\pi}\int_0^\infty dk\, k\sin(kF)\exp\Bigg\{ -\frac{4(2\pi)^{3/2} n k^{3/2}}{15} - \frac{8\pi q}{1+q}\int_0^\infty dx\, x^2 p(x)\left[1-\frac{x^2}{k}\sin\left(\frac{k}{x^2}\right)\right]\Bigg\}. \quad (22)$$

We limit the rest of the discussion to this isotropic case. As for the Poisson point process, it is not possible to find an explicit form of $P(\vec{F})$ [or $W(F)$]. However we can connect the asymptotic behaviors of $P(\vec{F})$ to those of $p(x)$ and to that of the Poisson case.

A. Large F expansion

In order to study the large F behavior of $P(\vec{F})$, it is important to use the general properties of the small k expansion of the characteristic function $A(\vec{k})$ to the lowest order greater than zero. In particular [16], in the isotropic case, if $P(\vec{F}) \simeq CF^{-\alpha}$ at sufficiently large F [or $W(F)\simeq 4\pi CF^{2-\alpha}$], with $C>0$ and $\alpha>3$ to guarantee $P(\vec{F})$ to be a normalized PDF in three dimensions, then

$$A(\vec{k}) = \int d^3F \exp(-i\vec{k}\cdot\vec{F})P(\vec{F}) = \begin{cases} 1-\frac{1}{6}\mathcal{F}^2k^2 & \text{if } \alpha>5 \\ 1-ak^{\alpha-3} & \text{if } 3<\alpha\leq 5, \end{cases} \quad (23)$$

where $\mathcal{F}^2=\int d^3F\, F^2 P(\vec{F})$ is the second moment of the force PDF, and $a>0$ is a constant characterizing the singular part of the small k expansion which is given by

$$a = 4\pi C\int_0^\infty dx\, x^{2-\alpha}\left(1-\frac{\sin x}{x}\right). \quad (24)$$

Note that $\alpha>5$ implies that $\mathcal{F}^2$ is finite. For the Poisson case $\alpha=9/2$ and $C=n/2$, and correspondingly, from Eq. (23), $A(\vec{k})\simeq 1-(4n/15)(2\pi k)^{3/2}$ at small k as it must be.

Therefore, our strategy consists in finding α and C by connecting the expansion given in Eq. (23) to the form of $p(x)$ and in particular to its small x behavior. Let us suppose that $p(x)\simeq Bx^\beta$ at sufficiently small x [in any case $B>0$ and $\beta>-3$ as $p(x)$ is a PDF of a three-dimensional stochastic variable]. It is quite simple to show that at small k the integral

$$I(k;\beta) = \int_0^\infty dx\, x^2 p(x)\left[1-\frac{x^2}{k}\sin\left(\frac{k}{x^2}\right)\right]$$

behaves as follows:

$$I(k;\beta)\simeq \begin{cases} c_1 k^{(3+\beta)/2} & \text{if } -3<\beta<1 \\ c_2 k^2 & \text{if } \beta\geq 1. \end{cases} \quad (25)$$

For $\beta=1$ there will be logarithmic corrections to Eq. (25). c_1 and c_2 are two positive constants depending on $p(x)$ in the following way:

$$c_1 = \frac{B}{2}\int_0^\infty dx\, x^{-(5+\beta)/2}\left(1-\frac{\sin x}{x}\right), \quad (26)$$

$$c_2 = \frac{1}{24\pi}\overline{\left(\frac{1}{x^4}\right)},$$

where $\overline{(\cdots)}=\int d^3x\overline{(\cdots)}p(x)$ is the average over the PDF $p(x)$. Consequently, by using the results of Eqs. (22)–(26), we can distinguish three cases for what concerns the asymptotic behavior of $P(\vec{F})$.

(1) For $\beta>0$ the dominating part in $A(\vec{k})$ at small k is exactly the same as in the homogeneous Poisson case with the same average density n, i.e.,

$$A(\vec{k})\simeq 1-\frac{4n}{15}(2\pi k)^{3/2}, \quad (27)$$

which implies $P(\vec{F})\simeq \frac{n}{2}F^{-9/2}$ [or equivalently $W(F)\simeq 2\pi nF^{-5/2}$] at large F with the same amplitude of the pure Poisson case. In fact in this case, as $\beta>0$, the shot noise at small distance from the particle in the origin, which we have seen to dominate the large F limit in the statistically homogeneous Poisson point process, is purely Poissonian receiving only a negligible contribution from $p(x)$.

(2) For $\beta=0$ we again have a scaling behavior typical of the isotropic Poisson case, but the coefficient C of $P(\vec{F})$ is larger, receiving a positive contribution from two-point correlations, i.e., from $p(x)$:

$$A(\vec{k})\simeq 1-8\pi n\left(\frac{(2\pi)^{1/2}}{15}+\frac{c_1 q}{n(1+q)}\right)k^{3/2}, \quad (28)$$

which implies again $P(\vec{F})\simeq CF^{-9/2}$ at large F but with a larger amplitude C than in the isotropic Poisson case:

$$C=\frac{n}{2}+\frac{qB}{1+q}. \quad (29)$$

In practice, from Eqs. (22)–(26), we have the same scaling behavior of $P(\vec{F})$ of the isotropic Poisson case but with a larger average density $n'=n+2qB/(1+q)$. This is due to the fact that the particle in the origin sees at small scales a spatial distribution locally identical to a Poisson one with such an effective average density.

(3) For $\beta<0$ the small k behavior of $A(\vec{k})$ is radically changed from the isotropic Poisson case, as the second term in Eq. (20) is dominant on $C_P(k)$. In particular from Eqs.(19), (20), and (25) we have that

$$a=\frac{8\pi c_1 q}{1+q},$$

i.e.,

$$A(\vec{k})\simeq 1-\frac{8\pi c_1 q}{1+q}k^{(3+\beta)/2}. \tag{30}$$

This means [see Eq. (23)] that $2-\alpha=-(5+\beta)/2$. From this relation we see that in this case the two integrals, respectively in Eq. (24) and in the first of Eq. (26) coincide. This implies that $P(\vec{F})\simeq CF^{-(9+\beta)/2}$ [or equivalently $W(F)\simeq 4\pi CF^{-(5+\beta)/2}$] with

$$C=\frac{q}{1+q}B. \tag{31}$$

This is due to the fact that at small scales the particle in the origin sees a strongly nonuniform, radially decreasing effective density of particles.

In all cases the constant C can be calculated as a function of B and β by using Eqs. (30) and (26).

B. Small *F* expansion

The small F behavior of $P(\vec{F})$ can be connected to the large k behavior of its Fourier transform. First of all we note that

$$\lim_{k\to+\infty} 4\pi I(k;\beta)=4\pi\int_0^{+\infty}dx x^2 p(x)=1.$$

This simple observation implies [see Eq. (22)] that for any GP point process, the asymptotically large k behavior of $A(\vec{k})$ is similar to that of the isotropic Poisson case with the same average density, but with an amplitude reduced by a factor $\exp(-2q/(1+q)$. Consequently, the small F behavior of $W(F)$ is the same of the homogeneous Poisson point process but with an amplitude reduced by the same factor $\exp[-2q/(1+q)]$, i.e.,

$$W(F)\simeq \exp\left(-\frac{2q}{1+q}\right)\frac{4}{3\pi}F_0^{-3}F^2, \tag{32}$$

where F_0 is given by Eq. (9).

V. COMPARISON WITH SIMULATIONS

The validity of these theoretical results is well supported by the statistical analysis of numerical simulations of two different kinds of GP point processes with two explicit choices of $p(x)$ (see Fig. 1), for which the PDF $W(F)$ of F is directly measured

(1) In the first case $p(x)$ is chosen to be simply a positive constant up to a fixed distance $x_0>0$ and zero beyond this distance:

$$p(x)=\begin{cases}\dfrac{3}{4\pi x_0^3} & \text{if } 0<r\leq x_0\\ 0 & \text{if } r>x_0,\end{cases} \tag{33}$$

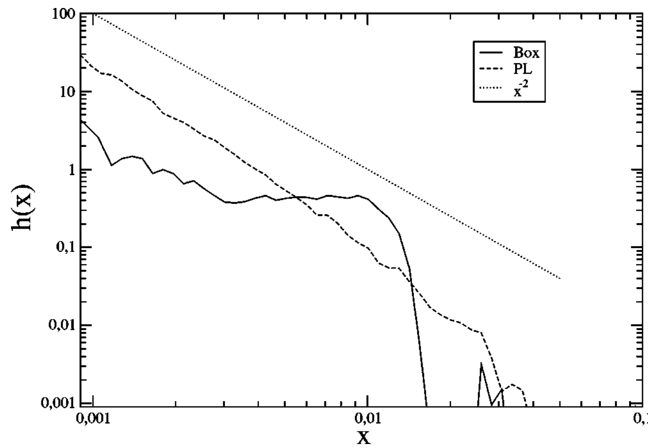

FIG. 1. Connected two-point correlation function $h(x)$ measured in a single realization, with 1.5×10^5 points in a cubic box of volume $V=1$ for the two Gauss-Poisson point processes both with $q=0.5$ and where, respectively, (i) $p(x)$ is the *box function* (box) given by Eq. (33) with cutoff at $x_0\simeq 0.012$ (continuous line), and where (ii) $p(x)=(1/4\pi x_0)\exp(-x/x_0)x^2$ (PL) with $x_0=0.012$ (dashed line). For comparison also the function $1/x^2$ is shown.

i.e., in reference to the preceding section $B=3/4\pi x_0^3$ and $\beta=0$. That is the probability of attaching a daughter particle at a distance between x and $x+dx$ from its parent is $3x^2dx/x_0^3$ if $x\leq x_0$ and zero for $x>x_0$, while the direction of $\vec{x}$ is completely random. As shown above this choice of $p(x)$ should give

$$W(F)\simeq\left(2\pi n+\frac{3q}{x_0^3(1+q)}\right)F^{-5/2}$$

at large F, that is, with the same exponent but with a larger amplitude than the pure isotropic Poisson case with the same average density *n*. At small F, as shown above, the asymptotic behavior of $W(F)$ should be given by Eq. (32).

(2) In the second case $p(x)$ decays exponentially fast at large x but it is singular as x^{-2} at small x, i.e.,

$$p(x)=\frac{1}{4\pi x_0}\frac{\exp\left(-\dfrac{x}{r_0}\right)}{x^2}. \tag{34}$$

This choice of $p(x)$ should give

$$W(F)\simeq\frac{q}{x_0(1+q)}F^{-(5+\beta)/2}$$

at large F with $\beta=-2$. Again Eq. (32) should be valid at small F.

The results of these simulations for the large and the small F scaling behaviors of $W(F)$ show a very good agreement (see Figs. 2 and 3) with the theoretical predictions given in the preceding section for what concerns both exponents and amplitudes. Consequently, the approximation at the base of

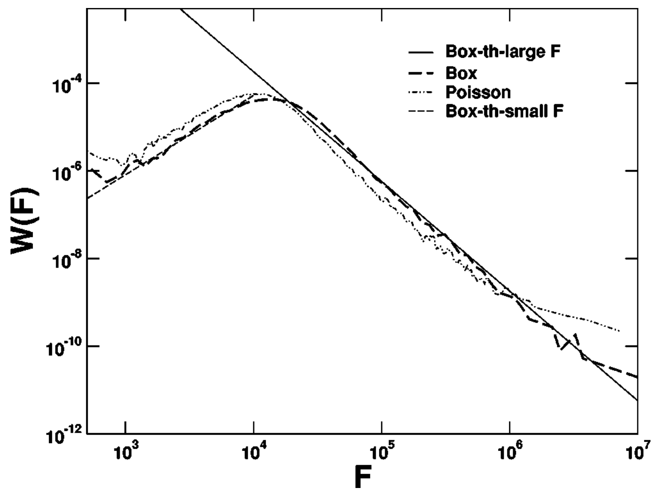

FIG. 2. Comparison between the theoretical predictions (Box-th-large F and Box-th-small F) given in the text and simulations (Box) of the PDF $W(F)$ of the modulus of the gravitational force for the GP case where $p(x)$ is a box function given by Eq. (33) with $q=0.5$ and $x_0 \simeq 0.012$, simulated with 1.5×10^5 point particles in a cubic volume of unit size. The theoretical behaviors at small and large fields computed as explained in the text show a very good agreement with simulations. For comparison the behavior of the Holtzmark PDF for a homogeneous Poisson particle distribution (Poisson) with the same number density is also shown (point-dashed line).

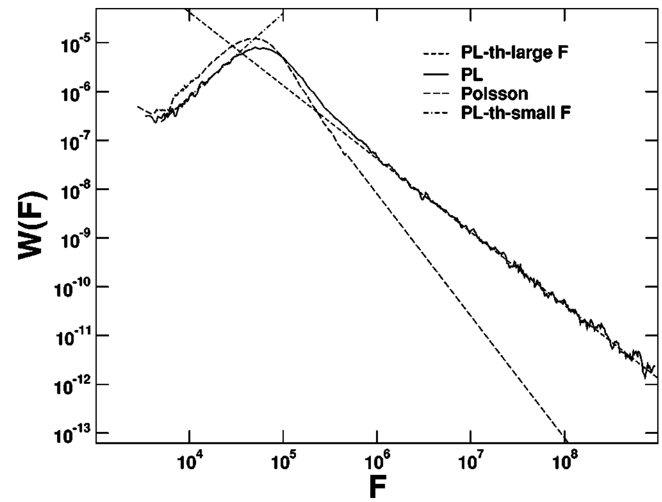

FIG. 3. As in the previous figure for the case where $p(x)$ is given by Eq. (34) with $q=0.5$ and $x_0 \simeq 0.0055$. The system has been simulated through 1.5×10^6 particles in a cubic volume of unit size (i.e., ten times the density in Figs. 1 and 2 but rescaling appropriately x_0) in order to increase the statistics to make more clear the modifications with respect the Poisson particle distribution (Poisson) with the same number density. Asymptotic theoretical predictions for small and large F are, respectively, indicated through PL-th-small F and PL-th-large F.

these calculations can be considered valid to study the one-point statistics of the gravitational field acting on the system particles.

VI. DISCUSSION

In this paper we have studied the gravitational force fluctuations in the so-called GP particle distributions, which can be considered as the class of most weakly correlated point processes beyond the Poisson one. For this particle system we have seen how to generalize, through appropriate approximations, the methods developed for the Poisson case in order to find the PDF of the gravitational force acting on each particle. In the GP case, important deviations from the Poisson behavior are found both in the small and in the large F limits. While in the former limit one has, with respect to the Poisson case, only a change of amplitude and a conservation of the scaling exponent, in the latter limit even the scaling exponent can be strongly modified. This can be caused mainly by the small scale behavior of CTPCF, which can introduce strong modification in the large force regime when diverging at small distances. All these theoretical predictions are confirmed by direct results in numerical simulations in which both the scaling exponents of the force PDF at large and small F and the amplitudes have been found in good agreement with the theoretical values obtained through the approximations used.

Before concluding, an important observation has to be made: as seen above in both Poisson and GP point processes, the contribution to the total gravitational force felt by a particle due to the other particles in its neighborhood is dominating in determining the PDF of the force (in particular in the large field limit). This implies that the gravitational force fluctuates a lot spatially from particle to particle. However this does not mean at all that the forces felt by two different particles are spatially uncorrelated. On the contrary it is simple to show that they are *strongly* correlated by analyzing the statistical information encrypted in the Poisson equation linking the gravitational field (i.e., force) $\vec{E}(\vec{x})$ in the spatial point $\vec{x}$ to the stochastic matter density $n(\vec{x})$ in the same point:

$$\vec{\nabla}\cdot\vec{E}(\vec{x}) = -n(\vec{x}). \tag{35}$$

By taking the ensemble average of the square modulus of the Fourier transform of both sides of Eq. (35) we obtain

$$\langle|\vec{k}\cdot\vec{E}_F(\vec{k})|^2\rangle = \langle|n_F(\vec{k})|^2\rangle, \tag{36}$$

where $\vec{E}_F(\vec{k})$ and $n_F(\vec{k})$ are, respectively, the Fourier transforms of $\vec{E}(\vec{x})$ and $n(\vec{x})$. The right-hand side of Eq. (36) is equal (for $k\neq 0$ and apart from a normalization factor $1/V$) to the power spectrum $S(\vec{k})$ of the density field which is the Fourier transform of the CTPCF $\xi(\vec{x})$. Since in a homogeneous Poisson point process $\xi(\vec{x})$ is given by Eq. (1), we have that $S(\vec{k})$ is positive and constant at all $\vec{k}$. Therefore, analyzing Eq. (36) for $k\rightarrow 0$, we can say that two-point correlations of the gravitational field decay in space at large separations x as $1/x$, i.e., very slowly. In the GP class of point processes the situation is analogous to the Poisson case, but the power spectrum $S(\vec{k})$ receives a contribution also from the nondiagonal part $h(\vec{x})$ of $\xi(\vec{x})$. Since in all GP point processes $h(\vec{x})$ is integrable over all the space, with positive integral, we obtain again that field-field correlation decays as $1/x$, but with a larger amplitude with respect to a Poisson particle distribution with the same average density. This shared behavior of Poisson and GP point processes is due to the combination of two facts.

(1) For both cases $\xi(\vec{x})$ has a finite and positive integral over all the space.

(2) The fact that the contribution to the gravitational field (i.e., force) in a point due to all faraway particles varies slowly in space, because of the long-range nature of the elementary particle-particle gravitational interaction (i.e., $\sim 1/x^2$).

Finally we can say that the importance of this work is twofold. (i) First, this is the first case of statistically homogeneous correlated particle distribution in which a systematic study of the gravitational force Chandrasekhar is done. (ii) This study suggests some basic ingredients to be used in future attempts of extending the analysis to more complex correlated particle distributions.

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